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INITIAL SEGMENTS OF THE DEGREES OF CONSTRUCTIBILITY

BY

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ABSTRACT

We show that any constructible, constructibly countable, (dual) algebraic lattice is isomorphic to the degrees of constructibility of reals in some generic extension of L.

Introduction

The motivating question for this paper, in its boldest form, is: What are the initial segments of the degrees of constructibility?

That question, as posed, is in a sense necessarily unanswerable: The degrees of constructibility are notoriously non-absolute. To circumvent this, it is reasonable to make some assumptions that guarantee a rich degree structure and to invoke some cardinality restrictions. (By analogy, assuming ZFC, any countable upper semi-lattice with least element is isomorphic to an initial segment of the Turing degrees; see [10], [11], [13], and [12] for historical highlights, and [2] for an extension to upper semi-lattices of size $\aleph_1$.) One might ask: Assume the set-theoretic universe is rich (e.g. ω_1 is inaccessible from reals). What structural properties suffice to characterize the countable initial segments of the degrees of constructibility of reals?

Unfortunately, the answer to this question (under any appropriate interpre-

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tation of "structural") is that there is no such characterization. (For example, some but not all countable well-founded distributive lattices, and countable complete linear orderings, are isomorphic to the degrees below some real; see [1].) The complication is that $\leqq_c$ ("constructible in") is a constructible relation; any set x constructs the ordering on the degrees below x. Thus, any structural property which allows an upper semi-lattice U to code an arbitrary real leaves open the possibility that U codes a real r such that the degrees below r are too complicated to be embedded in U. Such a U cannot be an initial segment of the degrees.

Restricting the question to constructible upper semi-lattices makes it somewhat more tractable. In particular, there is no coding problem; any real coded into a constructible upper semi-lattice is constructible. Thus, we ask: Given appropriate richness assumptions on the universe, characterize the countable upper semi-lattices in L which are isomorphic to initial segments of the degrees of constructibility of reals.

For some constructible upper semi-lattices U, it is possible to construct a forcing partial order in L, which adds a generic real g such that the degrees below g are isomorphic to U. Iterated Sacks forcing ([16], [6]) suffices in case U is a countable successor ordinal. A generalized notion of iteration does the job for countable complete linear orderings in L [9]; the canonical example is a "backwards" copy of a successor ordinal. Adapting the techniques of the Turing degree results, Adamowicz ([3], [4]) produces such forcing partial orders for any well-founded countable upper semi-lattice in L. (In all these cases, "countable" is a somewhat stronger restriction than necessary.)

In the negative direction, there is a good reason for the failure of techniques from the Turing degrees to produce a similarly strong result about degrees of constructibility: Lubarsky [15] has shown that if the degrees of constructibility form a countable lattice, then it must be complete.

In this paper, we extend the positive results to include countable complete lattices which are not necessarily well-founded or linear, but which are algebraic. (This is essentially the property which we need to combine the ideas of [13] with those of [9]. It includes all well-founded countable lattices and all complete countable linear orderings.) This still leaves open the question of countable, constructible, complete but non-algebraic lattices. We do however show that there is a class of complete but non-algebraic lattices which cannot be realized as an initial segment of the degrees of constructibility by the forcing methods used here and for the Turing degrees.

We will consider a countable algebraic lattice $\mathscr{L}$ in L, and construct in L a forcing partial order P such that if $\mathcal{G}$ is P-generic over M, the degrees of constructibility of reals in $L[\mathcal{G}]$ are isomorphic to the lattice $\mathcal{L}$. As usual in such arguments, we will be forcing with trees ordered by inclusion, and the generic filter $\mathcal{G}$ will be equivalent to the unique common branch g through the trees in $\mathcal{G}$.

The properties of $\mathcal{L}$ will be used to construct an appropriate representation Θ of $\mathcal{L}$. The basic idea is that for each element i of $\mathcal{L}$, we have an equivalence relation $\equiv_i$ (congruent modulo i) on Θ. These relations form a lattice of equivalence relations isomorphic to $\mathcal{L}$. (There is a natural lattice structure on equivalence relations on a specified set, given by $\equiv_i \geqq \equiv_j$ iff $\equiv_i$ refines $\equiv_j$.) Our forcing conditions will be trees of finite sequences from Θ, and the generic object g will be a function from ω to Θ. From g we will define a family of reals $\{h_i \mid i \in \mathcal{L}\}$ such that the degrees of constructibility of the h_i form (in the canonical way) a copy of $\mathcal{L}$. (The real h_i is defined from g, by retaining information about the values $g(n)$ only up to its $\equiv_i$ equivalence class.) Finally, we will show that every real in $M[g]$ is of the same degree as one of the h_i.

Since the relations $\equiv_i$ form a copy of $\mathcal{L}$ as a lattice of equivalence relations, if $i \leqq j$ in $\mathcal{L}$ (and only in that case), $\equiv_j$ is a refinement of $\equiv_i$. This makes it clear that h_j constructs h_i. It is also not hard to guarantee, by giving the forcing conditions appropriate splitting, that if $\equiv_j$ is not a refinement of $\equiv_i$, then h_j does not construct h_i. The difficulty is in guaranteeing that every real is of the same degree as some h_i.

In Section 1, we define algebraic lattices and (sequential) algebraic lattice representations, and prove that a lattice is algebraic iff it has an algebraic representation.

In Section 2, we define, from a sequential algebraic representation of a countable lattice $\mathcal{L}$, the associated forcing partial order P, and prove that P generically adds reals h_i, $i \in \mathcal{L}$, whose degrees form an isomorphic copy of $\mathcal{L}$.

In Section 3, we complete the proof by showing that in the forcing extension of M by P, every real has the same degree as some h_i.

1. Algebraic lattice representations

Much of the history of initial segment results for the Turing degrees, as well as most other reducibilities, has been intimately connected with the problem of finding the right sort of representation for the lattice that one wants to embed. (See Lerman [14] for an overview of the process.) An analysis of the known Turing degree constructions and that of [9] lead us to a list of desirable properties for a representation. Bill Lampe identified a subset of these properties ((1.0)–(1.6) below with or without (1.4)) as characterizing the algebraic lattices (a result of Grätzer and Schmidt [8], Corollary II.1) and supplied us with many equivalent definitions for this class of lattices. We then showed that it is possible to get all the properties we wanted in a representation of any algebraic lattice. The relevant definitions and proofs for the entire result are presented in a basically self-contained way in this section. One *warning* however should be given to the reader who may already be familiar with algebraic lattices: Our notions are dual to the common ones. This is standard in all uses of representations by equivalence relations for initial segment results in degree theory because of the way in which the degrees form an upper semi-lattice but not a lattice.

DEFINITION 1.1. Let $\mathscr{L}$ be a lattice with universe some cardinal κ and relations $\leqq$, $\vee$, and $\wedge$. We will use the letters i, j, k, s and t to denote elements of $\mathscr{L}$ and F, I, S, and T to denote subsets of $\mathscr{L}$.

An element i of $\mathscr{L}$ is *compact* if, for every $I \subset \mathscr{L}$ with $\bigwedge I \leqq i$, there is a finite $F \subset I$ such that $\bigwedge F \leqq i$.

An *algebraic lattice* is a complete lattice which is compactly generated, that is, every $i \in \mathscr{L}$ is the infimum of the compact elements above it.

A set Θ of functions from $\mathscr{L}$ into κ (whose elements we denote by the letters α, β, γ or δ) is an *usl representation* of $\mathscr{L}$ if the following conditions are always met: (We define $\alpha \equiv_i \beta$ iff $\alpha(i) = \beta(i)$.)

(1.0) *Zero*: $\alpha \equiv_0 \beta$.

(1.1) *Ordering*: $i \leqq j$ & $\alpha \equiv_j \beta \Rightarrow \alpha \equiv_i \beta$.

(1.2) *Non-ordering*: $i \not\leqq j \Rightarrow \exists \alpha, \beta \in \Theta$ ($\alpha \equiv_j \beta$ & $\alpha \not\equiv_i \beta$)

(1.3) *Join*: $(i \vee j = k)$ & $\alpha \equiv_i \beta$ & $\alpha \equiv_j \beta \Rightarrow \alpha \equiv_k \beta$.

Θ is a *positive usl representation* if it satisfies (1.1) and (1.3) but not necessarily (1.2).

We say that the representation is *complete* if

(1.4) *Completeness*: $i = \bigvee I$ & $\forall j \in I (\alpha \equiv_j \beta) \Rightarrow \alpha \equiv_i \beta$.

Note that completeness (1.4) implies the join property (1.3).

We say that the representation is compact if

(1.5) *Compactness*: $i = \bigwedge I$ & $\alpha \equiv_i \beta \Rightarrow \exists$ finite $F \subset I$ and an $\underline{i} = \bigwedge F$ such that $\alpha \equiv_{\underline{i}} \beta$.

We say that an usl representation is a *lattice representation* if

(1.6) *Full meet*: For every $\alpha, \beta \in \Theta$ and every i, j and k in $\mathscr{L}$ with $i \wedge j = k$ and $\alpha \equiv_k \beta$ there are γ_1, γ_2 and γ_3 in Θ such that $\alpha \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \gamma_3 \equiv_j \beta$.

We say that a representation Θ is homogeneous if

(1.7) *Full homogeneity*: For every finite $\Theta' \subset \Theta$ and every $\alpha_0, \alpha_1, \beta_0, \beta_3 \in \Theta$ such that $\forall i \in \mathscr{L}(\alpha_0 \equiv_i \alpha_1 \to \beta_0 \equiv_i \beta_3)$ there are β_1 and β_2 in Θ and $f_0, f_1, f_2 : \Theta' \to \Theta$ such that $f_m(\alpha_0) = \beta_m, f_m(\alpha_1) = \beta_{m+1}$ and

$$\forall \alpha, \beta \in \Theta' \forall i \in \mathscr{L}(\alpha \equiv_i \beta \to f_m(\alpha) \equiv_i f_m(\beta)), \qquad \text{for } m = 0, 1, 2.$$

For the sake of brevity we call a complete compact homogeneous lattice representation simply an *algebraic representation* (we call it *positive* if it need not satisfy (1.2)). For our construction we will consider only countable algebraic lattices for which we will need a sequential approximation to such a representation:

Let $\mathscr{L}_n$ be an increasing sequence of finite subusls of a countable lattice $\mathscr{L}$ with union $\mathscr{L}$. A nested sequence Θ_n of finite positive complete compact usl representations for $\mathscr{L}$ which are also full usl representations for the $\mathscr{L}_n$ is a *sequential algebraic lattice representation* of $\mathscr{L}$ if

(1.6′) *Meet*: For every $\alpha, \beta \in \Theta_n$ and every i, j and k in $\mathscr{L}_n$ with $i \wedge j = k$ and $\alpha \equiv_k \beta$ there are γ_1, γ_2 and γ_3 in Θ_{n+1} such that $\alpha \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \gamma_3 \equiv_j \beta$.

and

(1.7′) *Homogeneity*: For every $\alpha_0, \alpha_1, \beta_0, \beta_3 \in \Theta_n$ such that $\forall i \in \mathscr{L}$ $(\alpha_0 \equiv_i \alpha_1 \to \beta_0 \equiv_i \beta_3)$ there are β_1 and β_2 in Θ_{n+1} and $f_0, f_1, f_2 : \Theta_n \to \Theta_{n+1}$ such that for $m = 0, 1, 2$ $f_m(\alpha_0) = \beta_m$, $f_m(\alpha_1) = \beta_{m+1}$ and $\forall \alpha, \beta \in \Theta_n$ $\forall i \in \mathscr{L}(\alpha \equiv_i \beta \to f_m(\alpha) \equiv_i f_m(\beta))$.

It should be clear that if a countable lattice $\mathscr{L}$ has an algebraic representation it has a sequential one. (The only change for uncountable ones would be that we should allow the $\mathscr{L}_n$ and Θ_n to have any cardinality less than that of $\mathscr{L}$, i.e. κ.) We thus wish to prove the following:

THEOREM 1.2. *Every algebraic lattice $\mathscr{L}$ has an algebraic representation.*

The construction of the desired representation and the verification that it has all the desired properties is rather long and frequently tedious. Many of the details overlap with results in the literature. Because the homogeneity requirement (1.7) is not considered in the lattice theoretic literature there does not seem to be a ready reference that would short circuit most of the work. We have opted to give all definitions in full and to state all the lemmas. Any verifications that are included in Lerman [13] or [14] have been omitted. They are in all cases straightforward (as are nearly all the new ones).

LEMMA 1.3. (a) *If* $i, j \in \mathscr{L}$ *and* $i \not\leq j$, *then there is a compact* $s \in \mathscr{L}$ *with* $j \leq s$ *and* $i \not\leq s$.

(b) *If* i *and* j *are compact then so is* $i \wedge j$. *Thus the infimum of any finite set of compact elements is compact.*

PROOF. (a) As j is the infimum of all the compact elements above it, there must be such an s if $i \not\leq j$.

(b) This is immediate from the definition of compact.

We begin by building a complete compact usl representation Θ_0 for $\mathscr{L}$. For each compact s in $\mathscr{L}$ we put in two elements α_s and β_s defined as follows:

$$\alpha_s(i) = \begin{cases} 0 & \text{if } i = 0, \\ \langle 0, 3s+1 \rangle & \text{if } i \neq 0, \end{cases}$$

and

$$\beta_s(i) = \begin{cases} 0 & \text{if } i = 0, \\ \langle 0, 3s+1 \rangle & \text{if } i \neq 0, \text{ and } i \leq s, \\ \langle 0, 3s \rangle & \text{if } i \neq 0 \text{ and } i \not\leq s. \end{cases}$$

LEMMA 1.4. Θ_0 *is a complete compact usl representation for* $\mathscr{L}$.

PROOF. (1.0) follows immediately from the definition of Θ_0. Note next that if $\alpha, \beta \in \Theta_0$, $i \neq 0$ and $\alpha \equiv_i \beta$ then, for some compact s in $\mathscr{L}$, they are α_s and β_s (in some order) and $i \leq s$. (1.1) and (1.3) are thus clearly satisfied. For (1.2) consider $i \not\leq j$. By Lemma 1.3 there is a compact s with $j \leq s$ and $i \not\leq s$. α_s and β_s then fulfill the requirements of (1.3).

For the completeness of the representation consider a situation as in (1.4). We again know that α and β are α_s and β_s and that $j \leq s$ for every $j \in I$. Thus $i \leq s$ and so by definition $\alpha_s \equiv_i \beta_s$ as required in (1.4). Finally, to verify compactness we consider a situation as in (1.5). Again we have to deal with α_s and β_s with $i \leq s$. As s is compact, there is a finite subset F of I such that $\underline{i} = \wedge F \leq s$. It is then clear from the definition of α_s and β_s that they are congruent modulo $\underline{i}$ as required in (1.5).

We now wish to prove two lemmas that will allow us to inductively build up extensions of the representation Θ_0 to satisfy the requirements embodied in (1.6) and (1.7).

LEMMA 1.5. *If* Θ *is a complete compact usl representation of* $\mathscr{L}$, $i, j, k \in$

$\mathcal{L}$, $i \wedge j = k$, $\alpha, \beta \in \Theta$ and $\alpha \equiv_k \beta$, then there is a $\Theta' \supset \Theta$ which is also a complete compact usl representation of $\mathcal{L}$ and contains elements γ_1, γ_2 and γ_3 such that $\alpha \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \gamma_3 \equiv_j \beta$ as required in (1.6).

Proof. The crucial observation is that it suffices to consider the situation in which i, j and k are all compact. Suppose we have a $\Theta^* \supset \Theta$ that has the desired elements whenever i, j and k are compact. Consider then an arbitrary $i \wedge j = k$ in $\mathcal{L}$. Let S_i and S_j be the sets of compact elements above i and j respectively so that $\wedge S_i = i$, $\wedge S_j = j$ and $\wedge(S_i \cup S_j) = k$. As Θ is compact there is a $\underline{k} \in \mathcal{L}$ and finite $F_i \subset S_i$ and $F_j \subset S_j$ such that $\underline{k} = \wedge(F_i \cup F_j)$ and $\alpha \equiv_{\underline{k}} \beta$. Now by Lemma 3, $\underline{k}$, as well as $\underline{i} = \wedge F_i$ and $\underline{j} = \wedge F_j$, are all compact. As $\underline{i} \wedge \underline{j} = \underline{k}$, we have by assumption γ_1, γ_2 and γ_3 in Θ^* as required in (1.6) for $\underline{i}, \underline{j}$ and $\underline{k}$. As $i \leqq \underline{i}$ and $j \leqq \underline{j}$, it is clear using (1.1) that the same γ's work for our original i, j and k as desired.

We now assume that i, j and k are compact and that $\alpha \equiv_k \beta$. If $i \leqq j$ we can take all the γ's to be α. We thus assume that $i \nleqq j$. We choose new numbers w, x, y and z not appearing as values of any element of Θ and define the required γ's as follows:

$$\gamma_1(t) = \begin{cases} \alpha(t) & \text{if } t \leqq i, \\ w & \text{if } t \nleqq i, \end{cases}$$

$$\gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \leqq j, \\ x & \text{if } t \leqq i \ \& \ t \nleqq j, \\ y & \text{otherwise}, \end{cases}$$

$$\gamma_3(t) = \begin{cases} \beta(t) & \text{if } t \leqq j, \\ x & \text{if } t \leqq i \ \& \ t \nleqq j, \\ z & \text{otherwise}. \end{cases}$$

We let $\Theta' = \Theta \cup \{\gamma_1, \gamma_2, \gamma_3\}$. As $i \nleqq j$, it is clear that the γ's satisfy the congruences required in (1.6). We thus need only verify that (1.1)–(1.5) hold for Θ'.

Note 1.6. Before considering each clause note the following:
(i) If $\gamma_1 \equiv_t \delta$ for $\delta \in \Theta$ then $t \leqq i$ and $\delta \equiv_t \alpha$.
(ii) If $\gamma_2 \equiv_t \delta$ for $\delta \in \Theta$, then $t \leqq j$, i (and so $t \leqq k$) and $\delta \equiv_t \alpha$.
(iii) If $\gamma_3 \equiv_t \delta$ for $\delta \in \Theta$, then $t \leqq j$ and $\delta \equiv_t \beta$.

(iv) If $\gamma_1 \equiv_t \gamma_2$, then $t \leqq i \wedge j = k$ and $\gamma_1 \equiv_t \alpha$.
(v) If $\gamma_1 \equiv_t \gamma_3$, then $t \leqq i \wedge j = k$ and $\gamma_1 \equiv_t \alpha \equiv_t \beta$.
(vi) If $\gamma_2 \equiv_t \gamma_3$, then either $t \leqq i, j$ and $\gamma_1 \equiv_t \gamma_2 \equiv_t \alpha \equiv_t \beta$ or $t \leqq i$ but $t \not\leqq j$.

The verification for (1.0)–(1.3) is straightforward and standard.

Consider next a situation as in (1.4) with $\vee T = \underline{t}$ and $\gamma_m \equiv_t \delta$ for every $t \in T$, $m = 1, 2$ or 3 and $\delta \in \Theta'$. We consider each possible case:

(a) $\delta \in \Theta$

$m = 1$. By Note 1.6(i), $t \leqq i$ and $\delta \equiv_t \alpha$ for each $t \in T$. Thus $\underline{t} \leqq i$ and, as Θ satisfies (1.4), $\delta \equiv_t \alpha$. By the definition of γ_1 we thus have $\gamma_1 \equiv_t \delta$ as required.

$m = 2$. As for $m = 1$ using (ii) in place of (i).

$m = 3$. As for $m = 1$ with α replaced by β and (i) by (iii).

(b) $\delta = \gamma_p$ for $p = 1, 2$ or 3, $p \neq m$

$m = 1, p = 2$. By 1.6(iv) $t \leqq k$ for every $t \in T$ and so $\underline{t} \leqq k$. Thus by definition $\gamma_1 \equiv_t \alpha \equiv_t \gamma_2$.

$m = 1$, $p = 3$. Again by 1.6(v) $t \leqq k$ and $\alpha \equiv_t \beta$ for every $t \in T$. Thus $\underline{t} \leqq k$ and, as Θ satisfies (1.1), $\alpha \equiv_t \beta$ and so by definition $\gamma_1 \equiv_t \gamma_3$.

$m = 2$, $p = 3$. By 1.6(vi) $t \leqq i$ for each $t \in T$ and in addition we have two subcases to consider:

(i) $\forall t \in T (t \leqq j)$ and so $\underline{t} \leqq k$

and

(ii) $\exists t \in T (t \not\leqq j)$ and so $\underline{t} \not\leqq j$.

(i) As Θ satisfies (1.1), $\alpha \equiv_{\underline{t}} \beta$. Thus by definition $\gamma_2 \equiv_{\underline{t}} \gamma_3$ as required.

(ii) By our assumptions $\underline{t} \leqq i$ and so by definition $\gamma_2 \equiv_t \gamma_3$.

The verification of (1.5) is quite similar. Suppose $\wedge T = \underline{t}$ and $\gamma_m \equiv_{\underline{t}} \delta$. We consider for $m = 1, 2$ or 3 and $\delta \in \Theta'$ each possible case:

(a) $\delta \in \Theta$

$m = 1$. By 1.6(i), $\underline{t} \leqq i$ and $\delta \equiv_{\underline{t}} \alpha$. As Θ satisfies (1.5), $\delta \equiv_{t'} \alpha$ for some $t' = \wedge F$ ($\geqq \underline{t}$) for some finite $F \subset T$. By the definition of γ_1 and (1.1) we thus have $\gamma_1 \equiv_{\underline{t}} \delta$ as required.

$m = 2$. Exactly as for $m = 1$ with i replaced by k and 1.6(i) by 1.6(ii).

$m = 3$. As for $m = 1$ with α replaced by β, i by j and 1.6(i) by 1.6(iii).

(b) $\delta = \gamma_p$ for $p = 1, 2$ or 3, $p \neq m$

$m = 1, p = 2$. By 1.6(iv), $\underline{t} \leqq k$ and $\gamma_1 \equiv_t \alpha \equiv_t \gamma_2$. By the compactness of k we have a t' and a finite $F \subseteq T$ with $t' = \wedge F \leqq k$. The definition of γ_1 and γ_2 then tells us that $\gamma_1 \equiv_{t'} \gamma_2$ as required.

$m = 1, p = 3$. By 1.6(v), $\underline{t} \leqq k$ and $\alpha \equiv_{\underline{t}} \beta$. As Θ satisfies (1.5), $\alpha \equiv_{t'} \beta$ for a t' as above and so by definition $\gamma_1 \equiv_{t'} \gamma_3$.

$m = 2, p = 3$. By 1.6(vi) $\underline{t} \leqq i$ and we have two subcases to consider:

(i) $\underline{t} \leqq j$ and $\alpha \equiv_{\underline{t}} \beta$

and

(ii) $\underline{t} \not\leqq j$.

(i) As Θ satisfies (1.5), we have a t' as above with $\alpha \equiv_{t'} \beta$. By the compactness of k and (1.1) we may also assume that $t' \leqq k$. Thus by definition $\gamma_2 \equiv_{\underline{t}} \gamma_3$ as required.

(ii) By the compactness of i we have a t' as above with $t' \leqq i$ while $t \not\leqq j$ guarantees that $t' \not\leqq j$. Thus by definition again, $\gamma_2 \equiv_{\underline{t}} \gamma_3$ as required.

LEMMA 1.7. *If Θ is a complete compact usl representation for $\mathscr{L}$ and the notation is as in* (1.7), *then there is a complete compact usl representation $\Theta'' \supset \Theta$ for $\mathscr{L}$ containing elements β_1 and β_2 such that there are maps $f_m : \Theta' \to \Theta''$ as required in* (1.7).

We first need another fact about compactness.

LEMMA 1.8. *If Θ is a complete compact usl representation for $\mathscr{L}$ and $j = \vee\{i : \alpha \equiv_i \beta\}$, then j is compact.*

PROOF. By completeness $\alpha \equiv_j \beta$. Let S_j be the set of compact elements above j so that $j = \wedge S_j$. By compactness there is a finite $F \subset S_j$ and a $\underline{j} = \wedge F$ such that $\alpha \equiv_{\underline{j}} \beta$. By definition of j, we see then that $j = \underline{j}$ which is compact by Lemma 1.3 as required.

PROOF OF LEMMA 1.7. Let $j = \vee\{i : \alpha_0 \equiv_i \alpha_1\}$. By Lemma 8, j is compact. Let $\Theta' = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_p\}$. Let u and v be new numbers not appearing as values (or coordinates of values) of elements of Θ. We define new elements β_n for $n = 1, 2$ and $\gamma_{m,q}$ for $m = 0, 1, 2$ and $q = 2, \ldots, p$ as follows:

$$\beta_n(t) = \begin{cases} \beta_0(t) & \text{if } t \leqq j, \\ \langle n, u \rangle & \text{if } t \not\leqq j, \end{cases}$$

$$\gamma_{m,q}(t) = \begin{cases} \beta_m(t) & \text{if } \alpha_q \equiv_t \alpha_0, \\ \beta_{m+1}(t) & \text{if } \alpha_q \equiv_t \alpha_1, \\ \langle v, m, \alpha_q(t) \rangle & \text{otherwise.} \end{cases}$$

We first note that $\beta_m \equiv_j \beta_{m+1}$ for $m = 0, 1, 2$ as $\beta_1 \equiv_j \beta_2 \equiv_j \beta_0$ by definition and $\beta_0 \equiv_j \beta_3$ by the hypotheses of (1.7). Thus $\alpha_0 \equiv_t \alpha_1$ implies that $\beta_m \equiv_t \beta_{m+1}$ and so the $\gamma_{m,q}$ are well defined.

We now set $\Theta'' = \Theta \cup \{\beta_1, \beta_2\} \cup \{\gamma_{m,q} : m = 0, 1, 2 \text{ and } q = 2, \ldots, p\}$. That Θ'' satisfies (1.0)–(1.3) and (1.7) is verified in Lerman [13, Lemma 2.4] albeit with a rather different notation scheme. We now verify that Θ'' satisfies (1.4) and (1.5). The argument for adding on β_1 and β_2 is the same as for γ_1 in Lemma 5. We therefore replace Θ by $\Theta \cup \{\beta_1, \beta_2\}$ and consider the cases for verifying first (1.4) and then (1.5) for $\gamma_{m,q} \equiv_t \delta$ as in Lemma 5.

NOTE 1.9. *(a) If* $\gamma_{m,q} \equiv_t \delta$ and $\delta \in \Theta$, then either

(i) $\gamma_{m,q} \equiv_t \beta_m$ and $\alpha_q \equiv_t \alpha_0$ or

(ii) $\gamma_{m,q} \equiv_t \beta_{m+1}$ and $\alpha_q \equiv_t \alpha_1$.

(b) If $\gamma_{m,q} \equiv_t \gamma_{n,r}$ then either

(i) $\alpha_q \equiv_t \alpha_r \equiv_t \alpha_0$ and $\beta_m \equiv_t \beta_n$

(ii) $\alpha_q \equiv_t \alpha_r \equiv_t \alpha_1$ and $\beta_{m+1} \equiv_t \beta_{n+1}$

(iii) $\alpha_q \equiv_t \alpha_0$, $\alpha_r \equiv_t \alpha_1$ and $\beta_m \equiv_t \beta_{n+1}$ or the roles of m, q and n, r are interchanged here or

(iv) $\alpha_q \equiv_t \alpha_r$, neither are congruent to α_0 or α_1 and $m = n$.

We now consider a situation as in (1.4) with $\underline{t} = \vee T$ and $\gamma_{m,q} \equiv_t \delta$ for every $t \in T$.

(a) $\delta \in \Theta$: We may assume as before that T is closed under finite joins. By Note 1.9(a) we must be in case (i) or (ii) for each $t \in T$. By the ordering property (1.1) of the representation, if $t' \leqq t$ are in T and one case holds for t then it holds for t' as well. Thus one case or the other must hold for cofinally many and so all $t \in T$. As the analyses are the same in each case assume (i) holds for all $t \in T$. As Θ satisfies completeness (1.4), we know that $\alpha_q \equiv_{\underline{t}} \alpha_0$ and $\beta_m \equiv_{\underline{t}} \delta$ and so by definition $\gamma_{q,m} \equiv_{\underline{t}} \beta_m \equiv_{\underline{t}} \delta$ as required.

(b) $\delta = \gamma_{n,r}$: As in case (a) we may assume that one of the cases of Note 1.9(b) holds for cofinally many $t \in T$: If there is a $t \in T$ such that 1.9(b)(iv) fails for all $t' \geqq t$ then one of (i)–(iii) holds for each such t'. As in case (a) these properties are inherited downward in T and as T is closed under finite join one of them must hold for all $t \in T$. We divide the analysis up according to which one holds for cofinally many $t \in T$.

(i) Again as Θ satisfies (1.4) $\alpha_q \equiv_{\underline{t}} \alpha_r \equiv_{\underline{t}} \alpha_0$ and $\beta_m \equiv_{\underline{t}} \beta_n$. The definitions then tell us that $\gamma_{m,q} \equiv_{\underline{t}} \gamma_{n,r}$ as required.

(ii) The argument is the same as in case (i) replacing α_0 by α_1 and m, n by $m + 1$, $n + 1$.

(iii) Here we see that $\alpha_q \equiv_{\underline{t}} \alpha_0$, $\alpha_r \equiv_{\underline{t}} \alpha_1$ and $\beta_m \equiv_{\underline{t}} \beta_{n+1}$. Once again the definitions give the desired congruences.

(iv) In this case we must have $\alpha_q \equiv_{\underline{t}} \alpha_r$. Of course, if they were congruent to

α_0 or α_1 modulo $\underline{t}$ then they would be so congruent modulo every $t \in T$ for a contradiction. Thus the definitions again give the required result.

We now carry out a similar analysis for (1.5). We begin with $\underline{t} = \wedge T$ and $\gamma_{m,q} \equiv_t \delta$.

(a) $\delta \in \Theta$

(i) As Θ satisfies (1.5) (and (1.1)) there is a $\underline{t}'$ which is the infimum over a finite subset of T such that $\beta_m \equiv_{\underline{t}'} \delta$ and $\alpha_q \equiv_{\underline{t}'} \alpha_0$. We have the required congruence $\gamma_{m,q} \equiv_{\underline{t}'} \delta$ by definition.

(ii) As for (i), replacing α_0 by α_1 and β_m by β_{m+1}.

(b) $\gamma_{n,r} = \delta$

(i) As Θ satisfies (1.5) there is a $\underline{t}'$ as above such that $\alpha_q \equiv_{\underline{t}'} \alpha_r \equiv_{\underline{t}'} \alpha_0$ and $\beta_m \equiv_{\underline{t}'} \beta_n$. Once again, that suffices.

(ii) and (iii) are the same as (i) modulo alphabetic variations.

(iv) Again we have a $\underline{t}'$ such that $\alpha_q \equiv_{\underline{t}'} \alpha_r$. As $\underline{t} \leqq \underline{t}'$ and neither α_q nor α_r are congruent to α_0 or α_1 modulo $\underline{t}$, then neither are congruent modulo $\underline{t}'$. Yet again then definitions give the desired congruence modulo $\underline{t}'$.

Proof of Theorem 1.2. We begin with the complete compact usl representation Θ_0 for $\mathscr{L}$ supplied by Lemma 1.4. As the union of a nested chain of complete compact usl representations for $\mathscr{L}$ is clearly also a complete compact usl representation for $\mathscr{L}$, we can iterate applications of Lemmas 1.5 and 1.7 to close off with a final representation $\Theta \supset \Theta_0$ which satisfies all of (1.0)–(1.7) as required.

We will use all of the properties of the representation of $\mathscr{L}$ in our construction of an isomorphic initial segment of the degrees of constructibility. To see that this precise construction will not work for any wider class of lattices we show that properties (1.1)–(1.5) are enough to guarantee that $\mathscr{L}$ is algebraic.

Proposition 1.9. *Any complete lattice with a complete compact usl representation (that is, one satisfying* (1.0)–(1.5)) *is algebraic.*

Proof. In fact, each $i \in \mathscr{L}$ is the infimum, $\underline{i}$, of those j which are of the form $\vee\{k : \alpha \equiv_k \beta\}$ for α and β such that $\alpha \equiv_i \beta$: This infimum is clearly $\geqq i$. For the other direction, first note that by (1.4) $\alpha \equiv_j \beta$ for each such j, α and β. If $\underline{i} \not\leqq i$, there would be, by (1.2), α and β such that $\alpha \equiv_i \beta$ but $\alpha \not\equiv_{\underline{i}} \beta$. The j corresponding to this pair α, β would have to be $\geqq \underline{i}$ by its definition but this would contradict (1.1). Thus $i = \underline{i}$. As each of these j is compact by Lemma 8, $\mathscr{L}$ is algebraic.

In Section 4 we will use Proposition 1.9 to show that, regardless of the assumptions made on the Θ_n, if our forcing construction produces an initial segment of the constructibility degrees which is a lattice then that lattice must be algebraic. We will also show that other possible constructions (depending on the extent to which they resemble ours or any other known initial segment argument) can succeed only if $\mathscr{L}$ satisfies conditions somewhat weaker than being algebraic (semi-continuity or continuity).

2. The forcing notion, the distinguished reals and their order

In the next two sections, we assume that $\mathscr{L}$ is a countable algebraic lattice in L.

In this section, we construct from a sequential algebraic representation of $\mathscr{L}$ the forcing partial ordering P. P will generically add to L distinguished reals $\{h_i \mid i \in \mathscr{L}\}$. In this section as well, we prove that the $\leqq_c$-ordering on the h_i is the same as the lattice ordering on $\mathscr{L}$. In the following section, we will prove that the degrees of constructibility in a generic extension of L by P are exactly the degrees of the h_i.

NOTATION. In the following two sections, we will assume that $V = L$ and that the lattice $\mathscr{L}$ has a sequential algebraic representation Θ given by $\langle \mathscr{L}_n \mid n < \omega \rangle$ and $\langle \Theta_n \mid n < \omega \rangle$. We will reserve the symbols i, j, k for denoting elements of $\mathscr{L}$, and α, β, γ for elements of Θ. Recall that an element of Θ is a map from $\mathscr{L}$ to ω and that we define equivalence relations on Θ by $\alpha \equiv_i \beta$ iff $\alpha(i) = \beta(i)$. The properties of the representation Θ that we need are listed in Definition 1.1: zero (1.0), ordering (1.1), non-ordering (1.2), completeness (1.4), compactness (1.5), meet (1.6′), and homogeneity (1.7′).

REMARK. Most of these properties of Θ are used to show that the embedding taking i to the degree of constructibility of h_i preserves the appropriate structure of $\mathscr{L}$. For example, the meet property is used in Lemma 3.4 to show that if $i \wedge j = k$, then any real constructible from both h_i and h_j is also constructible from h_k. The homogeneity property guarantees that the structure of the forcing conditions permits the fusion arguments which show that this control of the degrees of constructibility in the generic extension is as tight as possible: i.e., that no extraneous degrees are added.

NOTATION. ${}^{<\omega}\Theta$ is the collection of finite sequences from Θ, ordered by extension. We reserve the symbols σ, τ, ρ, π, to denote elements of ${}^{<\omega}\Theta$. The concatenation $\sigma^\wedge\langle\alpha\rangle$ may be unambiguously denoted by $\sigma^\wedge\alpha$. We say that $\sigma \equiv_i \tau$ iff $|\sigma| = |\tau|$ and, for all $n < |\sigma|$, $\sigma(n) \equiv_i \tau(n)$.

We will force with subtrees of ${}^{<\omega}\Theta$ to add a generic function g from ω to Θ. We will use g to denote both such a function, and the term in the forcing language for the generic object. Similarly, we will use r, s, t and h_i to denote both terms for reals, and their realizations. h_i denotes the specific real defined from g whose degree of constructibility corresponds to the lattice element i, given by $h_i(n) = g(n)(i)$. If we need to specify the interpretation of a term t relative to a given generic g, we will write $(t)^g$; the interpretations of the term h_i relative to g and g' may be denoted h_i and h'_i respectively. The restriction of a function f on ω to an integer m will be denoted $f \upharpoonright m$.

REMARK. Occasionally we will speak from the perspective of the generic extension $L[g]$ rather than that of L; the context should eliminate any ambiguity.

DEFINITION 2.1. A *tree* is a downward-closed subset T of ${}^{<\omega}\Theta$ such that every element of T has incomparable extensions in T.

DEFINITION 2.2. A node $\sigma \in T$ *splits* in T iff σ has at least two immediate successors, $\sigma^\wedge\alpha$ and $\sigma^\wedge\beta$, in T.

DEFINITION 2.3. $L_n(T)$, the nth *splitting* level of T, is

$$\{\sigma \in T \mid \sigma \text{ splits in } T \text{ and } |\{\tau \mid \tau \subset \sigma \text{ and } \tau \text{ splits in } T\}| = n\}.$$

DEFINITION 2.4. The *forcing partial order* $(P, <)$ is defined by:

$$P = \{T \mid T \text{ is a tree with the properties (1)–(4) of Definition 2.7}\}.$$

and

$$S \leqq T \quad \text{iff} \quad T \supseteq S.$$

($S \leqq T$ means that S is a stronger condition than T; S extends or refines T.)

DEFINITION 2.5. (The generic reals) Suppose $\mathscr{G}$ is a generic filter on P. We define, as usual,

$$g = \cap\{T \mid T \text{ is in } \mathscr{G}\} \qquad (g \text{ is thus a function from } \omega \text{ to } \Theta),$$

and for $i \in \mathscr{L}$ we define

$$h_i(n) = g(n)(i) \qquad (h_i \text{ is thus a function from } \omega \text{ to } \omega).$$

REMARK. As usual, $L[g] = L[\mathscr{G}]$, and $T \in \mathscr{G}$ iff g is a branch through T.

DEFINITION 2.6. Suppose $\sigma \in L_n(T)$ and $\sigma^\wedge\alpha \in T$. We define $\sigma^-\alpha$ to be the unique extension of $\sigma^\wedge\alpha$ in $L_{n+1}(T)$. (If necessary to avoid ambiguity, we write $\sigma^-\alpha^T$.)

DEFINITION 2.7. The properties required of elements of P are as follows:

(1) Splitting: If σ is in $L_n(T)$ then the immediate successors of σ in T are $\{\sigma^\wedge\alpha \mid \alpha \in \Theta_n\}$.

(2) Congruence: Suppose σ is in $L_n(T)$, and α and β are in Θ_n. Then for all i in $\mathscr{L}_n$, if $\alpha \equiv_i \beta$ then $\sigma^-\alpha \equiv_i \sigma^-\beta$.

(This condition says that, for σ in $L_n(T)$ and i in $\mathscr{L}_n$, if two immediate successors of σ are congruent mod i, then their extensions up to $L_{n+1}(T)$ must respect this congruence. The effect of this requirement is that, if g and g' are two paths through T, $g \restriction m = \sigma^-\alpha$, and $g' \restriction m = \sigma^-\beta$, then $h_i \restriction m = h'_i \restriction m$. Thus h_i carries less information than g.)

(3) Uniformity 1: For all n, all nodes on $L_n(T)$ have the same length.

(4) Uniformity 2: If σ and τ are both in $L_n(T)$ then for all ρ, $\sigma^\wedge\rho \in T$ iff $\tau^\wedge\rho \in T$.

(These uniformity conditions guarantee that, in the situation described above, h_i truly carries less information than g; since T above $\sigma^-\alpha$ and T above $\sigma^-\beta$ are identical, it may well be that g above $\sigma^-\alpha$ and g' above $\sigma^-\beta$ are identical, in which case $h_i = h'_i$ but $g \neq g'$.)

REMARK. In the following definitions and lemmas, we present some notation and basic facts that will be useful in manipulating trees. Some lemmas which are either easy to prove or completely standard are stated without proof.

DEFINITION 2.8. $T \leqq_n S$ iff $T \leqq S$ and $T \supseteq L_n(S)$.

REMARK. $T \leqq S$ means that T is a "thinning out" of S; $T \leqq_n S$ means that T is obtained by thinning out S above $L_n(S)$.

DEFINITION 2.9. A *fusion sequence* is a sequence $\langle T_n \mid n \in \omega \rangle$ from P such that for all n, $T_{n+1} \leqq_n T_n$. Its *fusion* is the tree $\bigcap\{T_n \mid n \in \omega\}$.

REMARK. The fusion of a fusion sequence is a condition.

DEFINITION 2.10. If $\sigma \in T$, $(T)_\sigma$ is the tree $\{\tau \in T \mid \sigma \supseteq \tau \text{ or } \tau \supseteq \sigma\}$. A *restriction* of T to σ is any condition S contained in $(T)_\sigma$.

LEMMA 2.11. *For any condition T and node $\sigma \in T$, restrictions of T to σ exist.*

PROOF. The tree $(T)_\sigma$ is not a condition only because it has too much splitting; it is too fat to satisfy the splitting property. We can thin it out to a condition S, defined inductively on the splitting levels of S:

$$L_0(S) = L_0((T)_\sigma);$$

$$L_{n+1}(S) = \{\sigma^\frown \alpha^T \mid \sigma \in L_n(S) \text{ and } \alpha \in \Theta_n\}.$$

DEFINITION 2.12. The condition S given above is the *canonical restriction of T to σ.*

REMARK. There are many possible restrictions of T to σ (and no least one). $(T)_\sigma$ is not itself a condition, but we will sometimes speak as though it were, writing "$(T)_\sigma \leqq S$" for "$S \supseteq (T)_\sigma$", and "$(T)_\sigma \Vdash \varphi$" for "every restriction of T to σ forces φ" (equivalently, $T \Vdash$ "if $g \supseteq \sigma$, then φ".)

DEFINITION 2.13. If $\sigma \in L_n(T)$ and $S \leqq (T)_\sigma$, an *amalgamation* of S into T (above $L_n(T)$) is a condition $R \leqq_n T$ such that $(R)_\sigma \leqq S$. In particular, if $S \Vdash \varphi$, then $R \Vdash$ "if $g \supseteq \sigma$, then φ".

LEMMA 2.14. *For any such T, σ, and S, an amalgamation R of S into T above $L_n(T)$ exists.*

PROOF. Choose any node $\sigma^\wedge\tau \in L_n(S)$. By uniformity of T, for any $\pi \in L_n(T)$, if $\sigma^\wedge\tau^\wedge\rho \in S$ then $\pi^\wedge\tau^\wedge\rho \in T$; let R be (the downward closure of) $\{\pi^\wedge\tau^\wedge\rho \mid \sigma^\wedge\tau^\wedge\rho \in S \text{ and } \pi \in L_n(T)\}$. Clearly R satisfies the uniformity and congruence properties; by the choice of τ, because S satisfies the splitting property, R does as well. Therefore R is a condition.

REMARK. Again, there are many possible choices for an amalgamation.

LEMMA 2.15. *For any T, $\sigma \in L_n(T)$, and open dense subset D of P, there is a condition $R \leqq_n T$ such that $(R)_\sigma \in D$.*

PROOF. Choose $S \leqq (T)_\sigma$ in D, and let R be an amalgamation of S into T above $L_n(T)$.

LEMMA 2.16. *For any T, n, and open dense subset D of P, there is an $R \leqq_n T$ such that for all $\sigma \in L_n(R)$, $(R)_\sigma \in D$. (In particular, R forces g to meet D in one of the $(R)_\sigma$ for each $\sigma \in L_n(R)$.)*

PROOF. Apply Lemma 2.15 finitely many times.

LEMMA 2.17. *If* $\langle D_n \mid n \in \omega \rangle$ *is a sequence of open dense sets and* T *is a condition, then there is an* $S \leqq T$ *such that for all* n *and all* $\sigma \in L_n(S)$, $(S)_\sigma \in D_n$.

PROOF. S is the fusion of a fusion sequence $\langle T_n \mid n \in \omega \rangle$ in which $T_0 = T$ and T_{n+1} is obtained by applying Lemma 16 to T_n and D_n.

COROLLARY 2.18. *P is an Axiom A notion of forcing and so forcing with P preserves* $\aleph_1$ [5].

COROLLARY 2.19. *Let t be any term for a real in $L[g]$, i.e., for a function from ω to ω. There is an (open) dense set of conditions S such that for all $n < \omega$ and all $\sigma \in L_n(S)$, $(S)_\sigma$ decides the value of $t(n)$.*

DEFINITION 2.20. We say that any condition S as in Corollary 2.19 *finitizes* the term t.

REMARK. Every condition finitizes the terms h_i.

DEFINITION 2.21. If s and t are terms for reals in $L[g]$, t is called *s-absolute* iff the realization of t in $L[g]$ depends only on the realization of s.

REMARK. If t is s-absolute, then t is forced to be constructed from s by a specific procedure which is fixed in the ground model. One may assume that t denotes the μth real constructed from s, for some fixed ordinal μ.

LEMMA 2.22. *If $T \Vdash$ "$t \leqq_c s$", then there is an $S \leqq T$ and an s-absolute term r such that $S \Vdash$ "$t = r$".*

PROOF. T forces that, for some ordinal μ, t is the μth real constructed from s. Choose S to fix a value for μ, and r to denote "the μth real constructed from s".

DEFINITION 2.23. Suppose T finitizes i. A *t-split* in T is a pair of nodes σ and τ on the same level of T such that, for some integers x and y, $(T)_\sigma \Vdash$ "$t(x) = y$", and $(T)_\tau \Vdash$ "$t(x) \neq y$". We then say that σ and τ *t-split on x*. If $\sigma \equiv_i \tau$, we say this is a *t-split mod i*.

REMARK. If $S \leqq T$, and two nodes σ and τ of S form a t-split in T, then they also form a t-split in S.

REMARK. A t-split in T mod i is a pair of nodes σ and τ such that $(T)_\sigma$ and $(T)_\tau$ carry the same possibilities for h_i, but force different facts about t. The presence of (too many) t-splits mod i forces that t will contain information not recoverable from h_i, i.e., that t will not be constructible from h_i.

DEFINITION 2.24. $T \Vdash^* $"$t \leqq_c h_i$" iff T finitizes t and T has no t-splits mod i. ($\Vdash^*$ may be read "strongly forces"; Lemma 2.25 justifies this terminology.)

LEMMA 2.25. *$T \Vdash^*$"$t \leqq_c h_i$" iff there is an h_i-absolute term r such that $T \Vdash$"$t = r$".*

PROOF. First suppose $T \Vdash^*$"$t \leqq_c h_i$". (Essentially, this means that nodes in T which are congruent mod i force the same facts about t in T. Thus t can be read off from the way in which T finitizes t just using g mod i which is just h_i. More formally:)

Since T finitizes t, $T \Vdash$"[$t(x) = y$ iff $(T)_\sigma \Vdash$"$t(x) = y$", where σ is the initial segment of g in $L_x(T)$]". Also, $\sigma \equiv_i \tau$ iff, for all $z < |\tau|$, $\sigma(z)(i) = \tau(z)(i)$; and in this case, σ and τ cannot form a t-split in T. But if σ is an initial segment of g, then $\sigma(z)(i) = h_i(z)$; i.e., τ is congruent to an initial segment of g mod i iff, for all $z < |\tau|$, $\tau(z)(i) = h_i(z)$. But now, $T \Vdash$"[$t(x) = y$ iff $(T)_\tau \Vdash$"$t(x) = y$", where τ is any node in $L_x(T)$ such that, for all $z < x$, $\tau(z)(i) = h_i(z)$]". This gives a specific real constructed from h_i, which is forced by T to equal t.

Conversely, suppose $T \Vdash$"$t = r$", where r is h_i-absolute; and by way of contradiction, suppose T has a t-split mod i, say the pair of nodes σ and τ. By uniformity, there is a canonical isomorphism between $(T)_\sigma$ and $(T)_\tau$, taking $\sigma{}^\wedge\rho$ to $\tau{}^\wedge\rho$; this induces an isomorphism between the portions of P above $(T)_\sigma$ and $(T)_\tau$, so that if g is a generic branch through $(T)_\sigma$, its isomorphic image $g' = \bigcup\{\tau{}^\wedge\rho \mid g \supseteq \sigma{}^\wedge\rho\}$ is a generic branch through $(T)_\tau$.

Since g and g' are both branches of T, we have $(t)^g = (r)^g$ and $(t)^{g'} = (r)^{g'}$. Since $\sigma \equiv_i \tau$ (and above those initial segments, $g = g'$), $h_i = h'_i$, so $(r)^g = (r)^{g'}$. But since σ and τ formed a t-split, $(t)^g \neq (t)^{g'}$, a contradiction.

COROLLARY 2.26. *If T forces "$t \leqq_c h_i$", T may be extended to strongly force "$t \leqq_c h_i$". If $T \Vdash^*$"$t \leqq_c h_i$" and $S \leqq T$, then also $S \Vdash^*$"$t \leqq_c h_i$".*

REMARK. The following lemma completes the tasks of this section.

LEMMA 2.27. *If g is P-generic, $h_i \leqq_c h_j$ iff $i \leqq j$.*

PROOF. First suppose i is not below j, and show that h_i is not constructible from h_j. By Corollary 2.26, if h_i were constructible from h_j, there would be a condition T in the generic filter such that $T \Vdash^* $ "$h_i \leqq_c h_j$". It suffices, by the definition of $\Vdash^*$ to show that every condition T has an h_i-split mod j. Choose n large enough so that i and j are in $\mathscr{L}_n$. By the non-ordering property of Θ, there are α and β in Θ_n such that α is congruent to β mod j but not mod i. Let T be any condition, and σ any element of $L_n(T)$; say $|\sigma| = x$. Both $\sigma\hat{}\alpha$ and $\sigma\hat{}\beta$ are in T; if they are initial segments of generics g and g' respectively, then $h_i(x) = g(x)(i) = \alpha(i) \neq \beta(i) = g'(x)(i) = h'_i(x)$; so $\sigma\hat{}\alpha$ and $\sigma\hat{}\beta$ form an h_i-split mod j.

If $i \leqq j$, to show $h_i \leqq_c h_j$, it suffices to show that no condition has an h_i-split mod j (so every condition strongly forces "$h_i \leqq_c h_j$"); but this follows from the ordering property of Θ.

3. Initiality

In this section we complete the proof, by showing that P forces any real in the generic extension to be equiconstructible with one of the h_i.

Remark. Our strategy is first to show that, for any real t, there is a least j such that $t \leqq h_j$ and then to show that for this j, $h_j \leqq t$.

Definition 3.1. A strong t-split is a t-split with the properties:

(a) The split has the form $\sigma^-\alpha\hat{}\rho$ and $\sigma^-\beta\hat{}\rho$ for some sequence ρ.

(b) The node σ is in $L_{n+1}(T)$, and α and β are in Θ_n.

Remark. We are about to do some manipulating of trees using the meet and homogeneity properties of Θ; these properties say that, if certain objects exist in Θ_n, other things can be found in Θ_{n+1}. Clause (b) gives us enough room to use these properties. The proof of Lemma 3.4 is illustrative.

Lemma 3.2. *Suppose T finitizes t and $T \Vdash$ "t is not constructible from h_j".* Then T has strong t-splits mod j.

Proof. First, by Lemma 2.25, we know that every such T has t-splits mod j.

Next, we show that every such T has t-splits mod j with property (a). Let σ and τ be a t-split mod j, with common initial segment $\rho \in L_m(T)$. Say $(T)_\sigma \Vdash$ "$t(x) = y$" and $(T)_\tau \Vdash$ "$t(x) \neq y$". Since T finitizes t, possibly by extending σ and τ, we can assume they lie in $L_{m+n}(T)$, $m + n$ large enough so that for every $\pi \in L_{m+n}(T)$, $(T)_\pi$ decides the value $t(x)$. We can write σ as $\rho\hat{}\sigma(1)\hat{}\cdots\hat{}\sigma(n)$, where $\rho\hat{}\cdots\hat{}\sigma(z)$ is in $L_{m+z}(T)$, and τ similarly. Consider $\tau' = \rho\hat{}\sigma(1)\hat{}\tau(2)\hat{}\cdots\hat{}\tau(n)$; $\tau' \in T$ by uniformity. Either $(T)_{\tau'} \Vdash$ "$t(x) = y$"

(in which case, τ and τ' form a t-split mod j with property (a),) or else $(T)_{\tau'} \Vdash$ "$t(x) \neq y$" (in which case, σ and τ' form a t-split mod j with common initial segment in $L_{m+1}(T)$, and the argument can be repeated with τ' in place of τ; after at most n repetitions, a t-split with property (a) will be produced.)

Finally, we show that every such T has strong t-splits mod j. Choose $\rho \in L_m(T)$, for $m > 0$, and let R be the canonical restriction of T to ρ. Let σ and τ be a t-split mod j in R, with property (a) which split on x. Since T finitizes t, we may assume, possibly by extending σ and τ, that $(T)_\sigma$ and $(T)_\tau$ already decide $t(x)$, i.e., σ and τ form a t-split mod j in T, with property (a) (because of the canonical choice of R). We show they also have property (b) in T: Suppose their common initial segment is $\pi \in L_n(R)$; then $\sigma = \pi^-\alpha^\wedge\pi'$ and $\tau = \pi^-\beta^\wedge\pi'$ for α and β in Θ_n. But if $\pi \in L_n(R)$, then $\pi \in L_{n+m}(T)$; thus the pair σ and τ have property (b) in T.

Corollary 3.3. *If $T \Vdash$ "t is not constructible from h_j", then for any σ in T, T has a strong t-split* mod j *extending σ.*

Proof. In the third part of the proof of the above lemma, choose ρ to extend σ.

Lemma 3.4. *Suppose $k = i \wedge j$, $T \Vdash$ "$t \leqq_c h_i$", and $T \Vdash$ "$t \leqq_c h_j$". Then $T \Vdash$ "$t \leqq_c h_k$".*

Remark. That is to say, the (finite) meet structure of $\mathscr{L}$ also carries over to the degrees.

Proof. Suppose not. By extending T if necessary, we may assume that $T \Vdash$ "t is not constructible from h_k", T finitizes t, $T \Vdash^*$ "$t \leqq_c h_i$", and $T \Vdash^*$ "$t \leqq_c h_j$". Choose n large enough so that i, j, and k are in $\mathscr{L}_n$. By possibly increasing n and Corollary 3.3, we may choose a strong t-split mod k with common initial segment σ in $L_n(T)$, say $(T)_{\sigma^-\alpha^\wedge\rho} \Vdash$ "$t(x) = y$", and $(T)_{\sigma^-\beta^\wedge\rho} \Vdash$ "$t(x) \neq y$". Possibly by extending ρ, we may assume that for every $\pi \in T$ on the same level of this t-split, $(T)_\pi$ decides the value $t(x)$.

Now by property (b) of the definition of strong t-split, and the meet property of Θ, there are γ_m in Θ_n, $m = 0, 1, 2, 3, 4$, such that $\alpha = \gamma_0 \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \gamma_3 \equiv_j \gamma_4 = \beta$. Since $\sigma^-\alpha^\wedge\rho$ and $\sigma^-\beta^\wedge\rho$ split in T on x, so do $\sigma^-\gamma_m{}^\wedge\rho$ and $\sigma^-\gamma_{m+1}{}^\wedge\rho$ for some m; but this means that T has a t-split mod i or mod j, contradicting $T \Vdash^*$ "$t \leqq_c h_i$" or $T \Vdash^*$ "$t \leqq_c h_j$".

Lemma 3.5. *Suppose $\langle i(n) \mid \omega \rangle$ is a decreasing sequence (in the ground* *model L) of elements of $\mathscr{L}$ with infimum i and, for all n, $T \Vdash$ "$t \leqq_c h_{i(n)}$". Then $T \Vdash$ "$t \leqq_c h_i$".*

PROOF. Suppose, by way of contradiction, that $T \Vdash$ "t is not constructible from h_i".

For all m, choose $f(m)$ large enough so that, for α and β in Θ_m, if $\alpha \equiv_i \beta$, then $\alpha \equiv_{i(f(m))} \beta$. This is possible by the compactness property of Θ. Given a condition S, we say the property $p(m, S)$ holds iff, for all $\sigma \in L_m(S)$ and α and β in Θ_m, if $\alpha \equiv_{i(f(m))} \beta$ then $\sigma^-\alpha \equiv_{i(f(m))} \sigma^-\beta$. (Note that if $p(m, S)$ holds and $n \geqq m + 2$, then for $R \leqq_n S$, $p(m, R)$ holds as well. So, if a fusion sequence has the property that for all m, $p(m, T_{m+2})$ holds, then for all m, $p(m, T_\omega)$ holds.)

To obtain our desired contradiction, it suffices to find $S \leqq T$ such that for all m, $p(m, S)$ holds, and for $\sigma \in L_m(S)$, $(S)_\sigma \Vdash^*$ "$t \leqq_c h_{i(f(m))}$". To show this suffices, suppose S is such a condition. By Corollary 3.3, S has a strong t-split mod i, say $\sigma^-\alpha{}^\wedge p$ and $\sigma^-\beta{}^\wedge p$ for $\sigma \in L_m(S)$. Since $\alpha \equiv_i \beta$, by the choice of f, $\alpha \equiv_{i(f(m))} \beta$. But then by $p(m, S)$, $\sigma^-\alpha \equiv_{i(f(m))} \sigma^-\beta$. This means that $\sigma^-\alpha{}^\wedge p$ and $\sigma^-\beta{}^\wedge p$ form a t-split mod $i(f(n))$, which contradicts the assumption that $(S)_\sigma \Vdash^*$ "$t \leqq_c h_{i(f(n))}$".

We now build such an S. S will be the fusion of a sequence $\langle T_m \mid m < \omega \rangle$ such that for all m, $p(m, T_{m+2})$ holds and, for $\sigma \in L_m(T_m)$, $(T_{m+1})_\sigma \Vdash^*$ "$t \leqq_c h_{i(f(m))}$". Clearly this will do. Choose the T_m inductively, assuming for the induction hypothesis that in addition, $p(m, T_{m+1})$ holds for all m.

Begin by choosing an increasing sequence, $\langle d(m) \mid m < \omega \rangle$, such that for all m, $i(f(m)) \in \mathscr{L}_{d(m+1)}$ and a tree $T_0 \leqq T$ such that $T_0 \Vdash^*$ "$t \leqq_c h_{i(f(0))}$".

Extending T_n to T_{n+1}:

We assume $p(n-1, T_n)$ holds. For $\sigma \in L_n(T_n)$, there is, by Corollary 2.26, an $S \leqq (T_n)_\sigma$ such that $S \Vdash^*$ "$t \leqq_c h_{i(f(n))}$". We may choose $\sigma^\wedge\tau \in L_n(S)$ and extend T_n to

$$R = \{\pi^\wedge\tau^\wedge\rho \mid \pi \in L_n(T_n) \text{ and } \sigma^\wedge\tau^\wedge\rho \in S\}.$$

Clearly $R \leqq_n T_n$, and $(R)_\sigma \leqq S$. If $p(n-1, T_n)$ holds, then $p(n-1, R)$ holds (because for $\pi \in L_{n-1}(T_n)$, $\pi^-\alpha$ and $\pi^-\beta$ from T_n have been extended in exactly the same way to get $\pi^-\alpha$ and $\pi^-\beta$ in R). Thus, in finitely many steps, we may extend T_n to R such that $p(n-1, R)$ holds, and for each $\sigma \in L_n(R)$, $(R)_\sigma \Vdash^*$ "$t \leqq_c h_{i(f(n))}$".

Now to extend R to T_{n+1}, choose any $\sigma \in L_n(R)$ and $\sigma^\wedge\tau \in L_{d(n+1)}(R)$. Let $L_n(T_{n+1}) = \{\pi^\wedge\tau \mid \pi \in L_n(R)\}$; by the same reasoning that showed that $p(n-1, R)$ held, this guarantees $p(n-1, T_{n+1})$. By choice of $d(n+1)$,

$i(f(n)) \in \mathcal{L}_{d(n+1)}$. Since for $\rho \in L_n(T_{n+1})$, $\rho \in L_{d(n+1)}(R)$, it follows that, if $\alpha \equiv_{i(f(n))} \beta$ are in $\Theta_{d(n+1)}$, then $\rho^- \alpha^R \equiv_{i(f(n))} \rho^- \beta^R$. Thus if we let

$$L_{n+1}(T_{n+1}) = \{\rho^- \alpha^R \mid \rho \in L_n(T_{n+1}) \text{ and } \alpha \in \Theta_n\},$$

we will have $p(n, T_{n+1})$, as required for the induction hypothesis. Finally, we complete the definition of T_{n+1} by induction on splitting levels by setting

$$L_{m+1}(T_{n+1}) = \{\rho^- \alpha^R \mid \pi \in L_m(T_{n+1}) \text{ and } \alpha \in \Theta_m\} \qquad \text{for all } m \geqq n+1.$$

REMARK. This proof somewhat obscures the real idea here. In fact (guided by the fusion sequence we have built) we are building S to strongly force t to be constructible from h_i by the following procedure:

T_1 gives a procedure to recover t from $h_{i(n(0))}$, i.e. from g mod $i(n(0))$; but $n(0)$ was chosen so that, as far as the first splitting in T_1 is concerned, to know the path taken by g mod $i(n(0))$, it suffices to know g mod i. Once we know the first splitting, however, we are in some restriction of T_2 to a node on $L_1(T_2)$. This restriction gives a procedure to recover t from g mod $i(n(1))$; but, as before, to know the next splitting mod $i(n(1))$ it suffices to know g mod i. Once we know this splitting we are in a restriction of T_3, and so on. Thus at each stage, or each splitting level, to know g to the precision required to decide from S the next fact about t, it suffices to know h_i.

COROLLARY 3.6. *If t is any real in $L[g]$, then there is a least j such that h_j constructs t.*

PROOF. Let t be a real in $L[g]$, and $X = \{i \mid t \leqq_c h_i\}$. In $L[g]$, X is closed upward and, by Lemma 3.4, closed under finite meets. By Lemma 3.5, X is closed under infima of descending sequences in the ground model L. By Shoenfield absoluteness, $\mathcal{L}$ is complete in $L[g]$ and so the meet of X is defined. We want to show that the meet of X is in X, i.e., that X has a least element. Let j be the meet of X.

By Shoenfield absoluteness again, if i is a compact element of $\mathcal{L}$ in L, i remains compact in $L[g]$. Thus $J = \{i \mid j \leqq i \text{ and } i \text{ is compact}\}$ defines the same set in $L[g]$ and in L. If $i \in J$, by the definition of compactness, i is above some finite meet from X. Then, by the closure properties of X, $i \in X$. Thus $X \supseteq J$.

Finally, as $\mathcal{L}$ is algebraic, j is the meet of J. If j is the meet of finitely many elements from J, then by the closure of X under finite meets, j is in X. If not, since by Lemma 1.3, J is closed under finite meets, j can be written in L as the infimum of a descending sequence from J. By the closure of X under such infima, j is in X in this case as well.

REMARK. Our next, and last, major task is to show that every t is equiconstructible with some h_j; specifically, that t constructs the least h_j which constructs t.

In any setting where we are forcing with trees, the standard proof that $s \leqq_c t$ uses a fusion argument to build a condition T which strongly forces "$s \leqq_c t$"; i.e., any nodes of T that carry different information about s also carry different information about t. At stage n of the fusion argument, having chosen T_n, we need to guarantee that, for σ and τ in $L_n(T_n)$ carrying different information about s, $(T_{n+1})_\sigma$ and $(T_{n+1})_\tau$ force different facts about t. To do this, we want to extend σ to two nodes $\sigma(0)$ and $\sigma(1)$ carrying different information about some $t(x)$, extend τ to some node τ' which also decides $t(x)$ in T (hence which disagrees with one of the $\sigma(m)$) and then to replace $(T_n)_\sigma$ with $(T_n)_{\sigma(m)}$ and to replace $(T_n)_\tau$ with $(T_n)_{\tau'}$ in the extension T_{n+1}.

This is essentially what we want to do in this setting. However, things aren't quite so simple; we have to choose the appropriate extensions of σ and τ carefully in order to retain the congruence and uniformity properties in T_{n+1}. The next lemma isolates the technical result we need in order to do this.

LEMMA 3.7. *Suppose T finitizes t, $\sigma \in L_n(T)$ and α_0 and α_1 are in Θ_n. Let k be the greatest element of $\mathscr{L}$ such that $\alpha_0 \equiv_k \alpha_1$. (Such a k exists by the completeness and zero properties of Θ.) If there is a strong t-split* mod k *in T above $\sigma^\frown\alpha_0$, then there is an $S \leqq_{n+1} T$ such that the pair $\sigma^\frown\alpha_0$ and $\sigma^\frown\alpha_1$ form a t-split in S.*

PROOF. To determine S, we will choose for each $\alpha \in \Theta_n$ a sequence $\sigma(\alpha)$ such that (for some fixed $m > n$);

(i) $\sigma^\wedge\sigma(\alpha)$ is an extension of $\sigma^\wedge\alpha$ in $L_m(T)$.

(ii) For all $i \in \mathscr{L}_n$ and α and $\alpha' \in \Theta_n$, if $\alpha \equiv_i \alpha'$, then $\sigma(\alpha) \equiv_i \sigma(\alpha')$.

(iii) $\sigma^\wedge\sigma(\alpha_0)$ and $\sigma^\wedge\sigma(\alpha_1)$ are a t-split in T.

Then we define S inductively by splitting level:

$$L_{n+1}(S) = \{\tau^\wedge\sigma(\alpha) \mid \tau \in L_n(T) \text{ and } \alpha \in \Theta_n\}.$$

For $m' \geqq n+1$,

$$L_{m'+1}(S) = \{\tau^\frown\alpha^T \mid \tau \in L_{m'}(S) \text{ and } \alpha \in \Theta_{m'}\}.$$

By construction and clause (i), S satisfies the uniformity properties on conditions at level $n+1$; clause (ii) guarantees that S satisfies the congruence property there; and clause (iii), that $\sigma^-\alpha_0$ and $\sigma^-\alpha_1$ form a t-split in S. That the requirements for being a condition are met at levels above $n+1$ is immediate from the definition and their being satisfied in T.

Assume then that there is a strong t-split mod k in T above $\sigma^-\alpha_0$, say $\rho^-\beta_0{}^\wedge\pi$ and $\rho^-\beta_3{}^\wedge\pi$ t-split on x. For some $N>n$, $\rho\in L_N(T)$ and β_0 and β_3 are in Θ_{N-1}. Possibly by extending π, we may assume this split is in $L_m(T)$ for an m such that any restriction of T to a node in $L_m(T)$ decides the value $t(x)$.

Before defining $\sigma(\alpha)$ we introduce the following notations:

For $\alpha\in\Theta_n$, $\rho'(\alpha)$ is the final segment of $\sigma^-\alpha$ above σ.
π' is the final segment of ρ above $\sigma^-\alpha_0$.
For $\beta\in\Theta_N$, $\rho(\beta)$ is the final segment of $\rho^-\beta$ above ρ.

Now, our strong t-split has the form $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_0){}^\wedge\pi$ and $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_3){}^\wedge\pi$. A typical candidate for $\sigma(\alpha)$ will be $\rho'(\alpha){}^\wedge\pi'{}^\wedge\rho(\beta){}^\wedge\pi$ for some $\beta\in\Theta_N$. This will guarantee clause (i).

We now have β_0, β_3, α_0, α_1 in Θ_{N-1}. Since $\beta_0\equiv_k\beta_3$ and k is largest element of $\mathscr{L}$ such that $\alpha_0\equiv_k\alpha_1$, we can apply the homogeneity property of Θ to obtain β_1 and β_2 in Θ_N and f_0, f_1, f_2 mapping Θ_{N-1} into Θ_N such that, for $M=0$, 1 or 2, each f_M preserves $\equiv_i$ for all $i\in\mathscr{L}$, $f_M(\alpha_0)=\beta_M$, and $f_M(\alpha_1)=\beta_{M+1}$. Note that the restrictions of T to nodes $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ and $\sigma{}^\wedge\rho'(\alpha_1){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ decide the value $t(x)$.

Case 1: For some M, restrictions of T to $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ and $\sigma{}^\wedge\rho'(\alpha_1){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ decide $t(x)$ differently, i.e., those two nodes form a t-split in T. (This is the easy case.) Let $\sigma(\alpha)$ be $\rho'(\alpha){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$. Clause (iii) is then satisfied. Since all $\sigma(\alpha)$ and $\sigma(\alpha')$ are the same except for initial segments $\rho'(\alpha)$ and $\rho'(\alpha')$, which are the final segments in T of $\sigma^-\alpha$ and $\sigma^-\alpha'$, clause (ii) is immediate.

Case 2: For all M, restrictions of T to $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ and $\sigma{}^\wedge\rho'(\alpha_1){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ decide $t(x)$ the same way. For some fixed $M<3$, $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ t-splits in T on x with $\sigma{}^\wedge\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_{M+1}){}^\wedge\pi$, and hence also with $\sigma{}^\wedge\rho'(\alpha_1){}^\wedge\pi'{}^\wedge\rho(\beta_{M+1}){}^\wedge\pi$. Now let $\sigma(\alpha)$ be $\rho'(\alpha){}^\wedge\pi'{}^\wedge\rho(f_M(\alpha)){}^\wedge\pi$. In particular, $\sigma(\alpha_0)=\rho'(\alpha_0){}^\wedge\pi'{}^\wedge\rho(\beta_M){}^\wedge\pi$ and $\sigma(\alpha_1)=\rho'(\alpha_1){}^\wedge\pi'{}^\wedge\rho(\beta_{M+1}){}^\wedge\pi$. This guarantees clause (iii). To see that clause (ii) is satisfied: suppose $\alpha\equiv_i\alpha'$, and $i\in\mathscr{L}_n$. By the choice of f_M, $f_M(\alpha)\equiv_i f_M(\alpha')$. Since $i\in\mathscr{L}_n$, by the congruence property of the condition T, $\rho(f_M(\alpha))\equiv_i\rho(f_M(\alpha'))$; hence, $\sigma(\alpha)\equiv_i\sigma(\alpha')$.

This completes the proof.

LEMMA 3.8. *Suppose* $T \Vdash$ "j is the least element of $\mathcal{L}$ *such that* $t \leqq_c h_j$". *Then* $T \Vdash$ "$h_j \leqq_c t$".

PROOF. We may assume $T \Vdash^*$ "$t \leqq_c h_j$". We want to construct $S \leqq T$ such that whenever two nodes in S carry different information about h_j (i.e., are not congruent mod j) then they also carry different information about t (form a t-split in S). Thus S will strongly force, in the appropriate sense, "$h_j \leqq_c t$". It suffices to construct $S \leqq T$ such that, for any $\sigma \in L_n(S)$ and α not congruent to β mod j in Θ_n, the pair $\sigma^\frown\alpha$ and $\sigma^\frown\beta$ t-split in S on some value x and such that every restriction of S to a node in $L_{n+1}(S)$ decides $t(x)$.

First, we show that this suffices: Assume S has this property. Choose N large enough so $j \in \mathcal{L}_N$.

Suppose first that σ and τ are two nodes of S which are not congruent mod j and that σ and τ have a common initial segment in $L_N(S)$. We may assume that $\sigma \supseteq \rho^\frown\alpha$ and $\tau \supseteq \pi^\frown\beta$, where $|\rho| = |\pi|$, ρ is on or above $L_N(S)$, $\rho \equiv_j \pi$, but $\rho^\frown\alpha$ and $\pi^\frown\beta$ are not congruent mod j. (By uniformity $\rho^\frown\beta \equiv_j \pi^\frown\beta$, so $\rho^\frown\alpha$ and $\rho^\frown\beta$ are not congruent mod j. By the assumption $j \in \mathcal{L}_N$, this means that α and β are not congruent mod j.) Since $T \Vdash^*$ "$t \leqq_c h_j$" and $\rho^\frown\beta \equiv_j \pi^\frown\beta$, then $\rho^\frown\beta$ and $\pi^\frown\beta$ are not a t-split. But by assumption on S, $\rho^\frown\alpha$ and $\rho^\frown\beta$ t-split in S on some value x which is also decided by $(S)_{\pi^\frown\beta}$ (which necessarily agrees with $(S)_{\rho^\frown\beta}$). Thus $\rho^\frown\alpha$ and $\pi^\frown\beta$ (and hence their extensions σ and τ) t-split in S on x.

Now we know that if σ and τ are not congruent mod j and have a common initial segment in $L_N(S)$, then they form a t-split in S. Thus S forces t to construct h_j by the following procedure (with parameter π, the finite initial segment of g in $L_N(S)$): To find any given $h_j(y)$, choose $\sigma \supseteq \pi$ in S such that $|\sigma| > y$, and such that whenever $(S)_\sigma$ forces a value for some $t(x)$, that value agrees with the actual one. Then $\sigma \equiv_j g \restriction |\sigma|$, for if not, σ and $g \restriction |\sigma|$ would t-split on some value x, contradicting the choice of σ. So $h_j(y) = g(y)(j) = \sigma(y)(j)$.

Second, we construct such an S. S will be the fusion of a sequence $\langle T_n \mid n < \omega \rangle$ with $T_0 = T$. To guarantee that S has the correct properties, we will choose $T_{n+1} \leqq_{n+1} T_n$ such that for $\sigma \in L_n(T_n)$ and α_0 and α_1 in Θ_n not congruent mod j, $\sigma^\wedge\alpha_0$ and $\sigma^\wedge\alpha_1$ form a t-split in T_{n+1} on some x such that any restriction of T_{n+1} to a node on $L_{n+1}(T_{n+1})$ decides $t(x)$. To do this, extend T_n to T_{n+1} in finitely many stages, at each stage taking care of a given σ, α_0, and α_1, using Lemmas 3.7 and 2.15. (Lemma 3.7 is applicable because, if α_0 and α_1 are not congruent mod j, then we can let k be the greatest element of $\mathcal{L}$ such that

$\alpha_0 \equiv_k \alpha_1$ (such a k exists by the zero and completeness properties of Θ). k is then not $\geqq j$, so $T \Vdash$ "t is not constructible from h_k"; and so by Corollary 3.3, there is a strong t-split mod k above every node, in particular $\sigma^- \alpha_0$, in any extension of T).

This completes the proof.

4. Conclusion

THEOREM 4.1. *Suppose $V = L$, and $\mathscr{L}$ is a countable algebraic lattice. Then there is a partial order P such that $\Vdash_P$ "The degree of constructibility of reals are isomorphic to $\mathscr{L}$".*

PROOF. By Theorem 1.2, $\mathscr{L}$ has a sequential algebraic representation. Let P be the partial order of Definition 2.4. By Lemma 2.27, $\Vdash_P$ "The degrees of constructibility of the reals h_i are isomorphic to $\mathscr{L}$". By Corollary 3.6 and Lemma 3.8, $\Vdash_P$ "Every real is of the same degree as some h_i".

COROLLARY 4.2. *In a sufficiently rich model of set theory (e.g. one in which $(\omega_2)^L$ is countable), every constructible algebraic lattice which is countable in L is isomorphic to an initial segment of degrees of constructibility of reals.*

COROLLARY 4.3. *By relativizing the above result we see that in every sufficiently rich model of set theory (e.g. one in which $(\omega_1)^{L[r]}$ is countable for every real r), every countable algebraic lattice is isomorphic to a segment of the degrees of constructibility of reals.*

Our basic result shows that a countable lattice in L can be isomorphic to an initial segment of degres if it is algebraic. An algebraic lattice is by definition complete. On the other hand Lubarsky's result [15] shows that completeness is a necessary restriction. This leaves an obvious gap. The ideal theorem in this setting would completely characterize those countable upper semi-lattices in L which can be initial segments of the degrees.

We close with some examples that show that one cannot get all complete constructibly countable lattices as initial segments of the constructibility degrees by using a forcing notion like the ones used here or in the setting of the Turing degrees.

EXAMPLE 4.5. Let $\mathscr{R}$ be the lattice consisting of the elements a_n for $n \in \omega$, 0, 1, and b. 0 and 1 are, of course, the least and greatest elements of $\mathscr{R}$ respectively. The other order relations of $\mathscr{R}$ are just that $a_{n+1} < a_n$ for $n \in \omega$.

The join and meet operations in $\mathscr{R}$ are determined by requiring that $b \vee a_n = 1$ and $b \wedge a_n = 0$ for every $n \in \omega$.

Suppose we try to realize $\mathscr{R}$ as an initial segment of the constructibility degrees by using any forcing notion P such that we can decode the representative h_i of the degree corresponding to $i \in \mathscr{L}$ from some type of representation of $\mathscr{R}$ in a way that respects the order and join properties of the representation. In particular suppose that, uniformly in x, y, z and m, we can read off the value of $h_x(m)$ from that of $h_y(m)$ for each $x \leqq y$ in $\mathscr{R}$ and from those of $h_z(m)$ and $h_w(m)$ for each z and w in $\mathscr{R}$ such that $z \vee w = x$. Consider the real f given by $f(n) = h_{a_n}(n)$. It is clear from our assumptions on the relation between the representation of $\mathscr{R}$ and its realization in the forcing extension that f is constructible from h_{a_n} for every n. Thus f must be constructible. On the other hand, we can, by assumption, compute $g(n) = h_1(n)$ from $h_{a_n}(n) = f(n)$ and $h_b(n)$. Thus we would have g constructible in f and h_b and so in h_b alone for a contradiction.

The crucial property of $\mathscr{R}$ in this argument is semi (or join) continuity.

DEFINITION 4.6. A lattice $\mathscr{L}$ is *semi-continuous*, or *join continuous*, if for every $x \in \mathscr{L}$ and every downward directed $S \subseteq \mathscr{L}$, $x \vee \bigwedge S = \bigwedge\{x \vee s \mid s \in S\}$.

It is easy to see that any failure of semi-continuity in a countable complete lattice provides an example like the one above. One simply replaces the a_n by a downward cofinal sequence in S, 0 by $\bigwedge S$, b by $x \vee \bigwedge S$ and 1 by $\bigwedge\{x \vee s \mid s \in S\} > x \vee \bigwedge S$. It is also clear that the argument will work in a somewhat more general situation as far as the forcing notion is concerned. The dependencies embodied in the ordering and join properties of our representations can be much more general. As long as the reductions depend pointwise on only constructibly much information, albeit in a uniform way, the same argument will work. It thus seems quite unlikely that one could realize complete constructible lattices which are not semi-continuous as initial segments of the degrees of constructibility by any forcing argument like the known tree notions of forcing.

If one restricts attention to forcing notions tied even more closely to our representations and methods of proof then we can rule out an even wider class of lattices.

DEFINITION 4.7. Let $\mathscr{L}$ be a complete lattice. We say that *y is way above x*, $y \gg x$, if $\wedge I \leqq x$ implies that there is a finite $F \subseteq I$ such that $\bigwedge F \leqq y$. Thus x is compact iff $x \gg x$. $\mathscr{L}$ is *continuous* if every element is the infimum of the elements way above it.

This definition, which is, as our wont, the dual of the one in the basic source on continuous lattices [7], shows their close connection with algebraic lattices. Indeed the way above relation is often called relative compactness. To extend our counterexamples, however, we need a characterization of continuous lattices in terms of a distributivity law (like that defining semi-continuity) whose dual form can be found in Theorem 2.3 of Chapter II of [7].

PROPOSITION 4.8. *A complete lattice $\mathscr{L}$ is continuous iff for every family $\{I_m : m \in M\}$ of downward directed subsets of $\mathscr{L}$ the following distributivity property holds*:

$$(*)\qquad \bigvee\{\bigwedge I_m : m \in M\} = \bigwedge\{\bigvee\{f(m) : m \in M\} : f \text{ is a choice function for the family } I_m\}.$$

Suppose now that $\mathscr{L}$ is a complete but not continuous countable lattice. By the countability of $\mathscr{L}$ we may assume in the failure of the condition (*) of Proposition 4.8 given by non-continuity that M is ω, each I_m consists of a decreasing sequence of elements $i_{m,n}$ with infimum $\underline{i}_m$ and that the infimum over all choice functions there can be restricted to one over a countable set f_n of such functions. As the only posible way (*) can fail is for the right-hand side to be strictly larger than the left, we can also assume that the $f_n(m)$ are decreasing as functions of n. For notational convenience we set $\underline{i} = \bigvee i_m$, $k_n = \bigvee\{f_n(m) : m \in M\}$ and $\underline{k} = \bigwedge k_n$.

Next suppose that we have a forcing extension $L[g]$ given by an argument like the one presented above in which the degrees of constructibility are isomorphic to $\mathscr{L}$ with the isomorphism given by $i \mapsto h_i$. We wish to show for our contradiction that $h_{\underline{k}}$ is constructible in $h_{\underline{i}}$. Now for each $n \in \omega$, $h_{\underline{k}}(n)$ can by the ordering property of our assumed representation be read off from $h_{k_n}(n)$. By completeness this can be read off from the entire sequence $\langle h_{f_n(m)}(n) : m \in \omega\rangle$. On the other hand as the $f_n(m)$ are decreasing in n, the sequence $\langle h_{f_n(m)}(n) : n \in \omega\rangle$ is constructible in $i_{m,n}$ for every n. It is therefore constructible in $h_{\underline{i}_m}$. Assuming that a reasonable proof that the map sending $i \in \mathscr{L}$ to the degree of h_i is an isomorphism supplies us with a reasonable amount of uniformity, we could expect to be able to construct this sequence uniformly in $h_{\underline{i}_m}$ and so the sequence over m of such sequences would be constructible from

$h_{\underline{i}}$. This double sequence is, by the remarks above, sufficient to construct $h_{\underline{k}}$ for our desired contradiction.

Finally we would like to show that, if one forces with trees defined from a representation exactly as we have done here to produce a generic extension in which the degrees of constructibility are isomorphic to the lattice of equivalence relations used to define the forcing relation, then the lattice is in fact algebraic. To be precise we assume that we have a nested sequence of sets Θ_n of maps from a given lattice $\mathcal{L}$ into ω with equivalence relations i defined on them by $\alpha \equiv_i \beta$ iff $\alpha(i) = \beta(i)$ for i in $\mathcal{L}$. Let P be the forcing notion defined from $\Theta = \bigcup \Theta_n$ as in Definition 2.7 and let g be P-generic over L. We claim that if the degrees of constructibility in $L[g]$ are isomorphic to $\mathcal{L}$ via the map sending $i \in \mathcal{L}$ to the degree of h_i (where $h_i(m) = g_m(i)$), then $\mathcal{L}$ is algebraic. By Proposition 1.9 and the results of [15], it suffices to verify that Θ has properties (1.0)–(1.5).

We begin by noting that the entire development of Section 2 up to the last lemma (2.27) does not depend on any special properties of Θ. In this light the arguments for Lemma 2.27 show that $h_i \leqq_c h_j$ iff there are no pairs $\alpha, \beta \in \Theta$ such that $\alpha \equiv_j \beta$ but $\alpha \not\equiv_i \beta$. Similarly, if there were $\alpha \not\equiv_0 \beta$ in Θ then h_0 would not be constructible. Thus Θ must satisfy (1.0)–(1.2). Next consider a possible counterexample I, α and β to (1.4) (which implies (1.3)). Let $i = \bigvee I$. Another diagonalization argument using Corollary 2.26 and this counterexample to (1.4) would show that $\bigoplus\{h_j : j \in I\}$ is not constructible from h_i for our next contradiction. Finally, we consider a possible counterexample, I, α and β to (1.5). We can assume without loss of generality that I consists of a descending sequence i_n with infimum i. Consider the real h defined by $h(n) = h_{i_n}(n)$. As in previous arguments it is clear that h is constructible in every h_{i_n} since we already know that Θ satisfies the ordering property. We wish to show for our final contradiction that h is not constructible from h_i. Once again we suppose it is and apply Lemma 2.26 and a diagonal argument. We have a tree T such that $T \Vdash^*$ "$h \leqq_c h_i$" and indeed by Lemma 2.25 an absolute term $\mu(h_i)$ such that $T \Vdash^*$ "$h = \mu(h_i)$". Let σ be a node in T at a level n large enough so that our counterexamples α and β belong to Θ_n. As $\alpha \equiv_i \beta$ we can choose a ρ such that $(T)_{\sigma^\frown\alpha^\frown\rho}$ and $(T)_{\sigma^\frown\beta^\frown\rho}$ force the same value for $\mu(h_i)(|\sigma|)$. On the other hand, as $\alpha \not\equiv_{i_n} \beta$ for every n $(T)_{\sigma^\frown\alpha^\frown\rho}$ and $(T)_{\sigma^\frown\beta^\frown\rho}$ force different values for $h(|\sigma|)$ for our desired contradiction.

It thus seems unlikely that current techniques can close the gap between our positive results and the negative ones of [15] by showing that any wider classes of lattices can be isomorphic to initial segments of the degrees of constructibility. One must look either to improvements in the negative direction or entirely new techniques for controlling initial segments of constructibility.

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